Multiscale methods applied to topology optimization: elasticity equation

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TOP Webinar 6

Contents

Topology Optimization and the Finite Element Method

Topology Optimization formulation

$$\begin{cases} \min_{\rho} \; J(\rho, u), \\ \text{subject to} \\ \qquad \mathcal{L}_{\rho}(u) = 0, \quad u \in \mathcal{U}_{ad}, \\ \qquad g_i(\rho, u) \leq 0, \quad i = 1, \dots, N_g, \\ \qquad \rho \in \mathcal{D}_{ad}, \end{cases}$$

Here $\mathcal{L}_{\rho}$ is a differential operator that models the physics.
Several issues:

- Penalization for black and white densities
- filtering
- MMA
- ...

Heat and Elasticity problem

	Heat conduction (2D)	Elasticity (2D)
Mass conservation law	$\operatorname{div}(\vec{q}) = f$	$\operatorname{div}(\sigma) = \vec{F}$
Constitutive law	$\vec{q} = -\mathbf{K}\nabla u$	$\sigma = -E\varepsilon(u)$
State equation	$-\operatorname{div}(\mathbf{K}\nabla u) = f$	$-\operatorname{div}(E\,\varepsilon(u)) = \vec{F}$
Weak form	$\int_\Omega (\mathbf{K}\nabla u)\nabla v = \int_\Omega fv$	$\int_\Omega (E\,\varepsilon(\vec{u})) : \varepsilon(\vec{v}) = \int_\Omega \vec{F} : \vec{v}$
Galerkin formulation	$\sum_{i\in I_\Omega} \int_\Omega K\ (\alpha_i \nabla\varphi_i)\,\nabla\varphi_j = \int_\Omega f\varphi_j$	$\sum_{i\in I_\Omega} \int_\Omega E\,(\alpha_i\varepsilon(\varphi_i))\,\varepsilon\varphi_j = \int_\Omega \vec{F}\varphi_j$
Matrix formulation	$A\vec{\alpha} = \vec{b}$	$\mathbf{A_E}\vec{\alpha} = \overrightarrow{b_E}$

TableComparison of stationary heat conduction and elasticity equations in two dimensions.

A density optimization problem

Continuous problem

$$\begin{cases} \min_{\rho} \; G(u_K(\rho), \rho) = \int_\Omega K(\rho)|\nabla u|^2 = \int_\Omega f u, \\ \text{subject to} \\ \qquad -\operatorname{div}(K(x)(\nabla u)) = f, \\ \qquad \int_\Omega \rho \, dV - V^* \leq 0, \\ \qquad 0 \leq \rho \leq 1. \end{cases}$$

Discretized problem

$$\begin{cases} \min_{\rho} \; c(\rho) = f^\top u, \\ \text{subject to} \\ \qquad \mathbf{A}_H(\rho) u = f, \\ \qquad V(\rho) - V^* \leq 0, \\ \qquad 0 \leq \rho \leq 1. \end{cases}$$

Minimum Compliance Design

Continuous problem

$$\begin{cases} \min_{\rho} \ G(u_K(\rho), \rho) = \int_\Omega F : u \, d\,\Omega + \int_\Gamma \sigma_n : u \, d\,S \\ \text{subject to} \\ \qquad -\operatorname{div}(C(x)(\varepsilon(u))) = F, \\ \qquad \int_\Omega \rho \, dV - V^* \leq 0, \\ \qquad 0 \leq \rho \leq 1. \end{cases}$$

Discretized problem

$$\begin{cases} \min_{\rho} \ c(\rho) = f^\top u, \\ \text{subject to} \\ \qquad \mathbf{A_E}(\rho) u = f, \\ \qquad V(\rho) - V^* \leq 0, \\ \qquad 0 \leq \rho \leq 1. \end{cases}$$

Problem complications

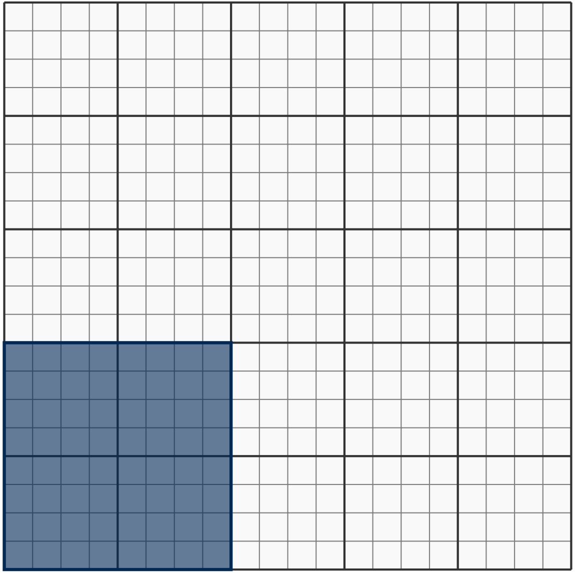
- Solve a FEM problem per iteration.
- Size of the problem.
- K has a multiscale structure, i.e. a complicated design.
- If $K_{\min} = E_0 \approx 0$ and $K_{\max} = 1$, we have a large contrast $\eta = \dfrac{K_{\max}}{K_{\min}}$.
- Around 99% of computing cost is due to the FEM method.

Finite Element Solves

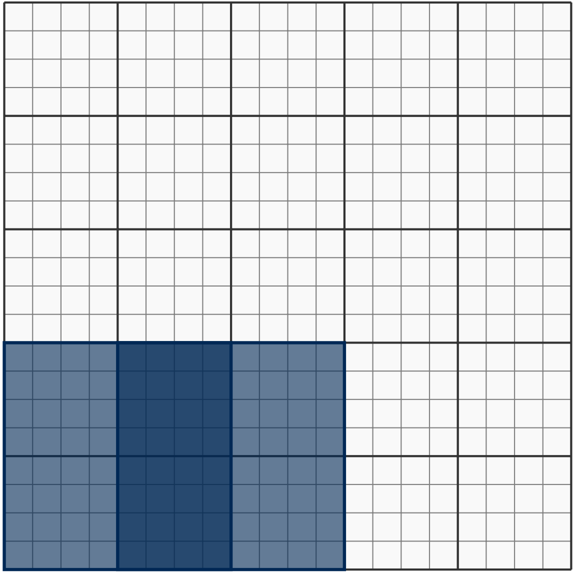
- Use an iterative solver for the linear problem $Au = b$, e.g. CG.
- The condition number depends on η and on the small scale variations of K.
- Build a preconditioner M^{-1}, with condition number independent of contrast and small scale variations. We solve $M^{-1}Au = M^{-1}b$

DD and Ms

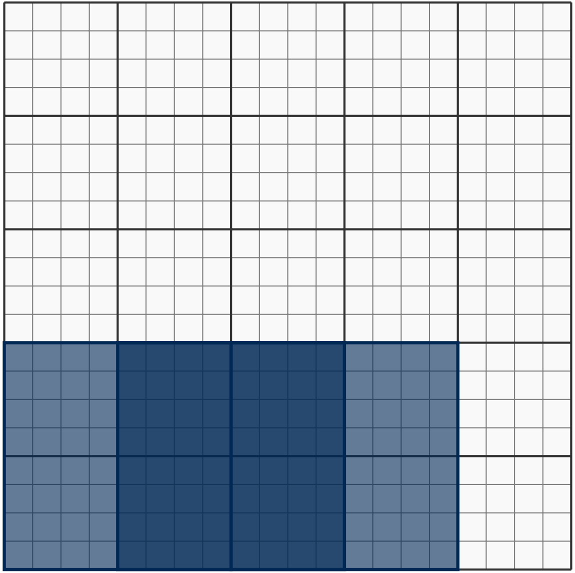
Domain Decomposition



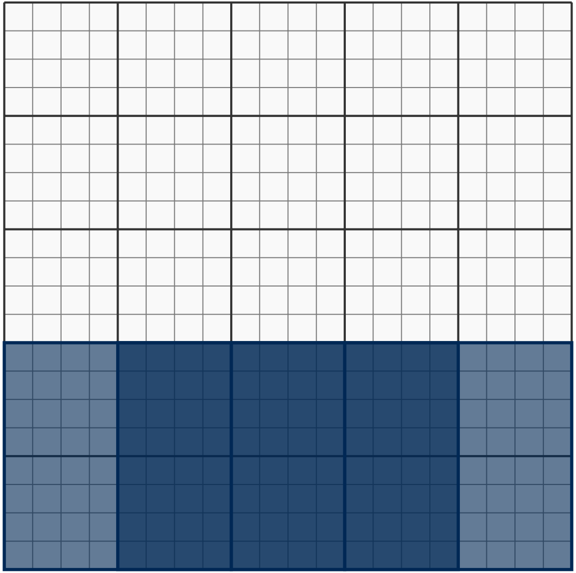
Domain Decomposition



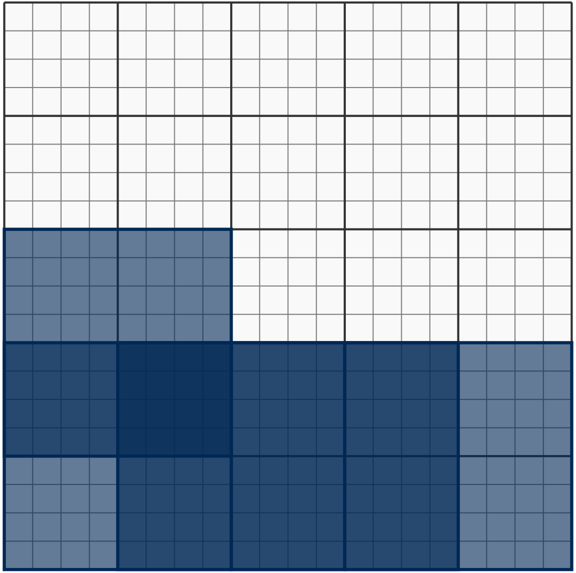
Domain Decomposition



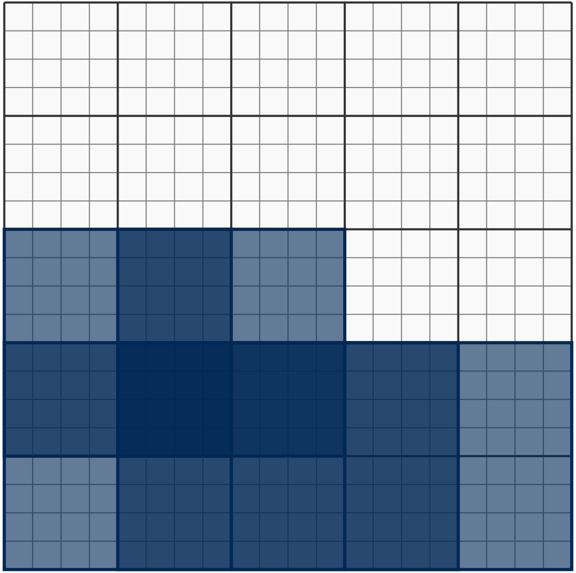
Domain Decomposition



Domain Decomposition



Domain Decomposition



Domain Decomposition

We add the local solutions to get u,

$$u = \sum_{i=1}^{N} R_i u_i,$$

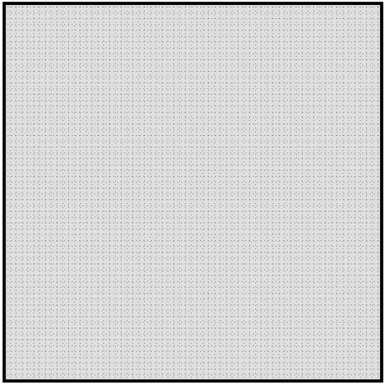
where $R_i : V_i \to V$ is the extension operator and $u_i = A_i^{-1} b_i$.
Then, the first level or the preconditioner is,

$$M_1^{-1} = \sum_{i=1}^{N} R_i A_i^{-1} R_i^T$$

where $A_i = R_i A_i R_i^T$

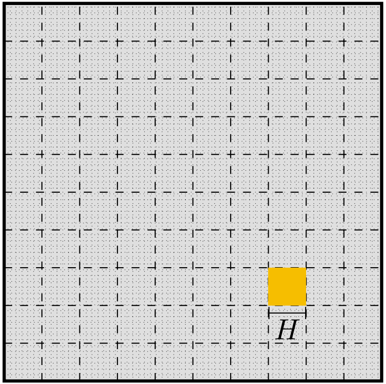
Multiscale method

$$Au = b, \quad \text{with } A \in \mathbb{R}^{N \times N}$$



Multiscale method

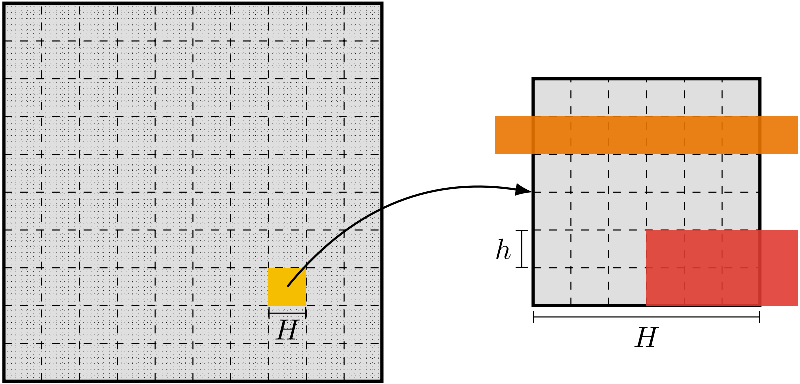
$$Au = b, \quad \text{with } A \in \mathbb{R}^{N \times N}$$



Multiscale method

$$Au = b, \quad \text{with } A \in \mathbb{R}^{N\times N}$$

$$A_0 u_0 = b_0, \quad \text{with } A_0 \in \mathbb{R}^{n\times n} \text{ and } n \ll N.$$



Multiscale method

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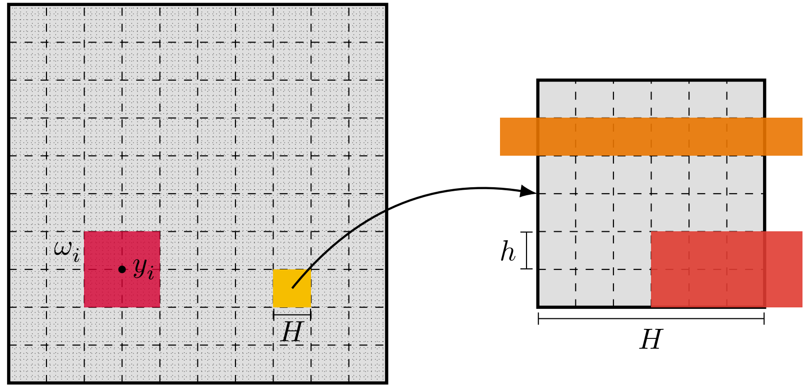


GMsFEM basis for the heat equation

We have,

$$A_0 u_0 = b_0,$$

with $A_0 = R_0 A R_0^\top$ and $b_0 = R_0 b$.
Where $R_0^\top$ is a downscaling operator that converts coarse-space coordinates into fine-grid space coordinates,

$$R_0^\top = [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_{N_t}].$$

As,

$$u \approx R_0^\top u_0,$$

we have the coarse level preconditioner,

$$M_2^{-1} = R_0 A_0^{-1} R_0^\top$$

GMsFEM basis for the heat equation

The coarse basis functions are obtained by an eigenvalue problem

$$-\mathrm{div}(\kappa(x)\nabla\psi_\ell^{\omega_i}) = \lambda_\ell^{\omega_i}\,\kappa(x)\,\psi_\ell^{\omega_i}.$$

Which in discrete form is

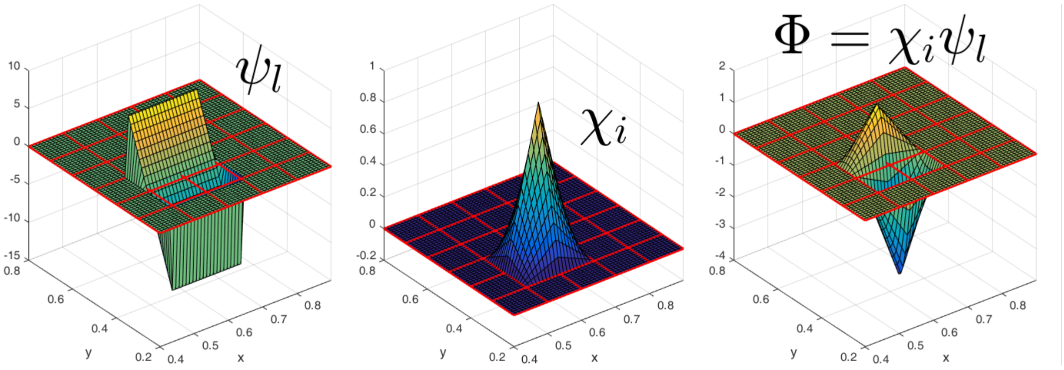
$$A^{\omega_i}\psi^{\omega_i} = \lambda^{\omega_i}M^{\omega_i}\psi^{\omega_i}.$$

For each ω_i we choose the number of basis by ordering the eigenvalues and choosing the ones below some threshold.
The basis functions are of the form

$$\chi_i\psi_\ell$$

where $\{\chi_i\}$ is a partition of unity.

GMsFEM basis for the heat equation



Coarse space for the elasticity equation

Eigenvalue Problem

$$-\operatorname{div}(C(x)\varepsilon(u)) = \lambda \begin{bmatrix} \kappa(x) & 0 \\ 0 & \kappa(x) \end{bmatrix} u, \qquad x \in \omega_i,$$

where $\boxed{C(x) = \kappa(x) C^0}$.
And in matrix form,

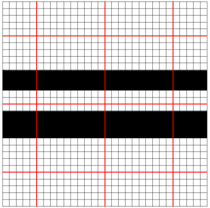
$$A_E^{\omega_i} \psi^{\omega_i} = \lambda M^{\omega_i} \psi^{\omega_i}, \tag{1}$$

The basis functions are of the form

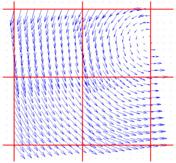
$$\chi_i \psi_\ell$$

where $\{\chi_i\}$ is a partition of unity. The number of contrast-dependent eigenvalues depends on the number of stiff-regions components.

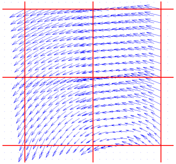
Illustration of important modes



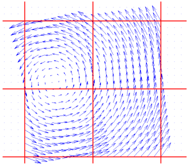
Coefficient with two channels



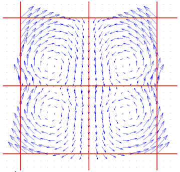
Mode 1



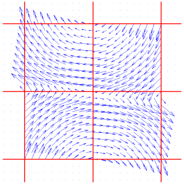
Mode 2



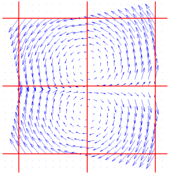
Mode 3



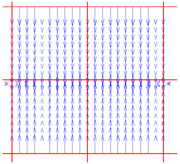
Mode 7



Mode 4

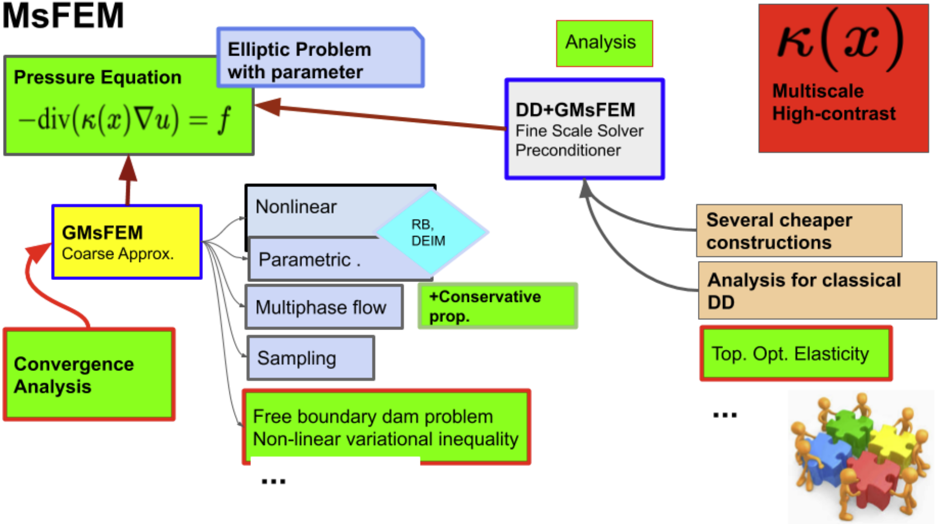


Mode 5



Mode 6

GMsFEM


Elliptic Problem with parameter
Pressure Equation
$-\mathrm{div}(\kappa(x)\nabla u) = f$
Analysis
$\kappa(x)$
Multiscale High-contrast
DD+GMsFEM
Fine Scale Solver Preconditioner
GMsFEM
Coarse Approx.
Nonlinear
RB, DEIM
Parametric .
Multiphase flow
+Conservative prop.
Sampling
Several cheaper constructions
Analysis for classical DD
Top. Opt. Elasticity
Convergence Analysis
Free boundary dam problem
Non-linear variational inequality
...
...

Two-levels Domain Decomposition

The first level, from Domain Decomposition is:

$$M_1^{-1} = \sum_{i=1}^{N} R_i A_i^{-1} R_i^\top,$$

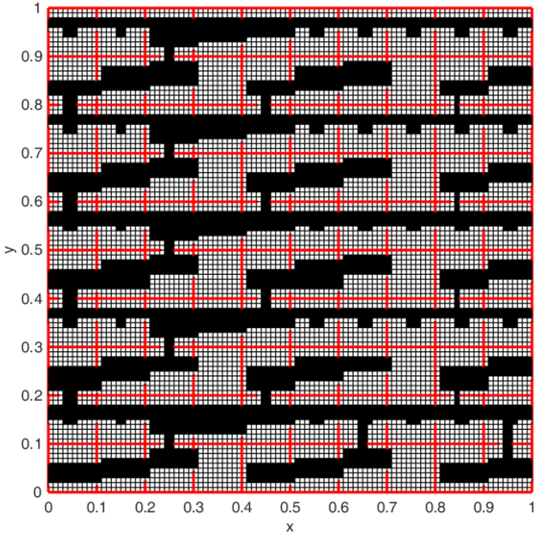
and the coarse level, from Multiscale is:

$$M_2^{-1} = R_0 A_0^{-1} R_0^\top,$$

which gives us the preconditioner:

$$M^{-1} = M_1^{-1} + M_2^{-1}.$$

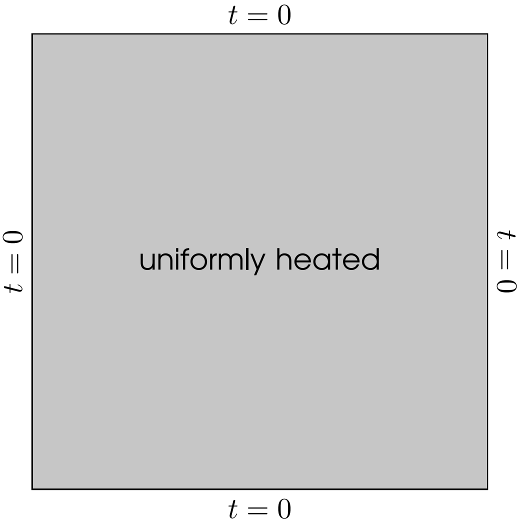
High-contrast coefficient example



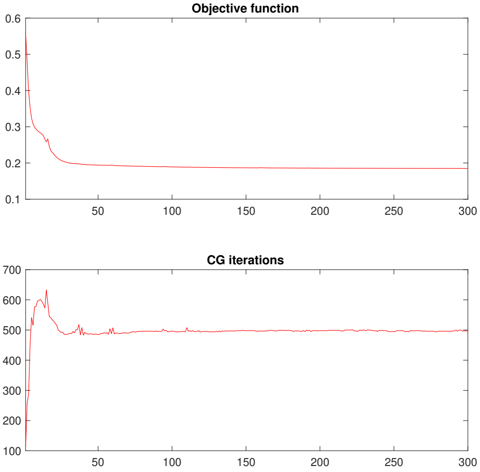
High-contrast coefficient example

Contrast	Iterations		Spectral condition	
	No Prec	Prec	No prec	Prec
1	112	17	2.0×10^3	4.8
1×10^{-2}	768	25	1.7×10^4	9.5
1×10^{-4}	2961	29	3.8×10^5	15.0
1×10^{-6}	7760	30	3.7×10^7	15.0

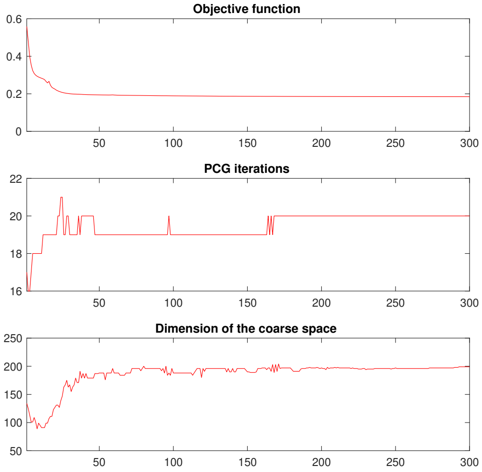
Topology optimization for the heat equation



Results



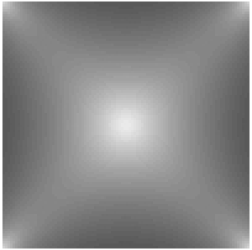
No preconditioner



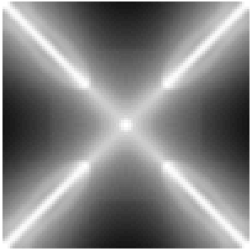
Two-levels preconditioner

Evolution of the design

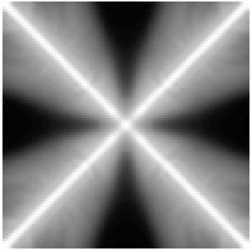
It: 001



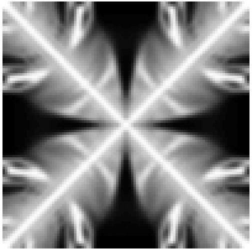
It: 005



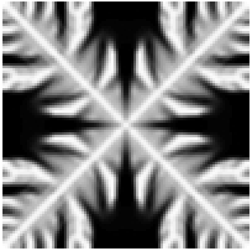
It: 010



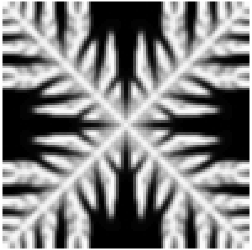
It: 015



It: 020



It: 030



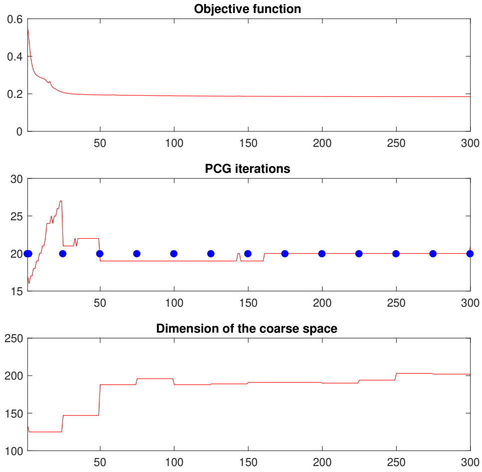
It: 150

It: 300

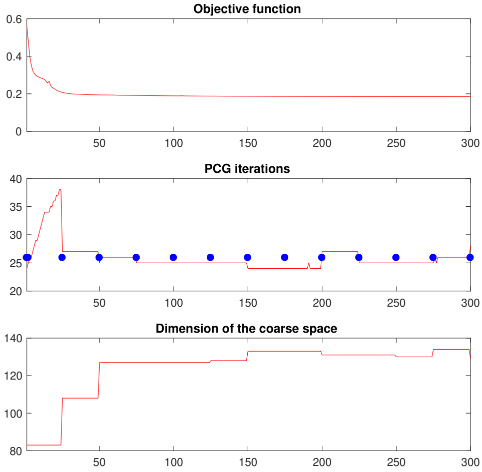
Coarse basis reuse

- For the first topology optimization iterations, calculate the coarse basis.
- If you do not like the last PCG iteration count, recompute the basis.

Results with coarse basis reuse



Two-levels preconditioner with reuse



Randomized option with reuse

Lower cost preconditioners

Elasticity and Heat

- Gustafsson, I., & Lindskog, G. (1998). On parallel solution of linear elasticity problems: Part I: theory. Numerical linear algebra with applications, 5(2), 123-139.
- Gustafsson, I., & Lindskog, G. (2002). On parallel solution of linear elasticity problems. Part II: Methods and some computer experiments. Numerical linear algebra with applications, 9(3), 205-221.

$$\begin{bmatrix} M_H^{-1} & 0 \\ 0 & M_H^{-1} \end{bmatrix} A_E u_E = \begin{bmatrix} M_H^{-1} & 0 \\ 0 & M_H^{-1} \end{bmatrix} .b_E.$$

Elasticity and Heat

- Gustafsson, I., & Lindskog, G. (1998). On parallel solution of linear elasticity problems: Part I: theory. Numerical linear algebra with applications, 5(2), 123-139.
- Gustafsson, I., & Lindskog, G. (2002). On parallel solution of linear elasticity problems. Part II: Methods and some computer experiments. Numerical linear algebra with applications, 9(3), 205-221.

$$\begin{bmatrix} M_H^{-1} & 0 \\ 0 & M_H^{-1} \end{bmatrix} A_E u_E = \begin{bmatrix} M_H^{-1} & 0 \\ 0 & M_H^{-1} \end{bmatrix} .b_E.$$

Low cost preconditioners

Enriching the heat equation coarse space enriched with rotations

$$M_{E,1}^{-1} + \begin{bmatrix} R_{0,rot} \\ R_{0,rot} \end{bmatrix} A_{0,E}^{-1} \begin{bmatrix} R_{0,rot} \\ R_{0,rot} \end{bmatrix}^\top \tag{2}$$

where

$$A_{0,E} = \begin{bmatrix} R_{0,rot} \\ R_{0,rot} \end{bmatrix} A_E \begin{bmatrix} R_{0,rot} \\ R_{0,rot} \end{bmatrix}^\top .$$

and $R_{0,rot}$ is constructed using the **Heat equation coarse basis** including the rotations:

- $[\chi_i \psi_\ell, 0]$
- $[0, \chi_i \psi_\ell]$
- $[\chi_i \psi_\ell r_1, \chi_i \psi_\ell r_2]$

Randomization

- ...
- Halko, N., Martinsson, P. G., & Tropp, J. A. (2011). Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. SIAM review, 53(2), 217-288.

Before computing, project into a small (sufficiently random/large) subspace.

Randomized eigenvectors approximation

(1) Generate forcing terms $f_1, f_2, \dots, f_M$ randomly ($\int_{\omega_i} f_\ell = 0$);
(2) Compute the local solutions $-\text{div}(K\nabla u_\ell) = f_\ell$ for heat and $-\text{div}(C\varepsilon(u_\ell)) = f_\ell$ for elasticity.
(3)Consider the matrix U_i whose columns generate the subspace $W_i = \text{span}\{u_\ell\} \cup \{1, r_x, r_y\}$. Then we can introduce the reduced size matrices,

$$\widetilde{\mathbf{A}_\mathbf{E}}^{\omega_i} = U_i^T \mathbf{A}_\mathbf{E}^{\omega_i} U_i, \quad \text{and } \widetilde{\mathbf{M}}^{\omega_i} = U_i^T \mathbf{M}^{\omega_i} U_i.$$

Then, we can solve the smaller dimension eigenvalue problem

$$\widetilde{\mathbf{A}_\mathbf{E}}^{\omega_i} \widetilde{\psi}^{\omega_i} = \tilde{\lambda}^{\omega_i} \widetilde{\mathbf{M}}^{\omega_i} \widetilde{\psi}^{\omega_i}. \tag{3}$$

Consider the approximations of the eigenvalues as

$$\lambda^{\omega_i} \approx \tilde{\lambda}^{\omega_i}, \tag{4}$$

and the approximation of the eigenvectors as,

$$\psi^{\omega_i} \approx U_i \widetilde{\psi}^{\omega_i}. \tag{5}$$

Numerical experiments

Results

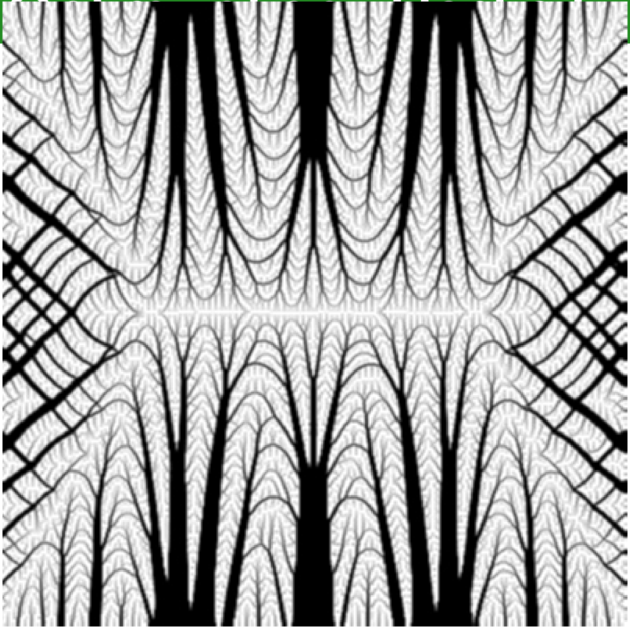
	Forcing term A			Forcing term B		
Coefficient	Iterations	Spectral condition	$\dim A_0$	Iterations	Spectral condition	$\dim A_0$
Without preconditioner	>2000	8.6×10^5	—	>2000	1.2×10^6	—
Two-levels elasticity preconditioner	62	114.6	387	53	113.8	387
Two-levels heat preconditioner	64	111.8	387	56	140.8	387
Two-levels heat random preconditioner	72	112.5	387	62	111.4	387
Two-levels elasticity random preconditioner	68	124.3	387	58	122	387

TableResults for the elasticity problem with high contrast coefficient $\kappa = 1 \times 10^{-4}$

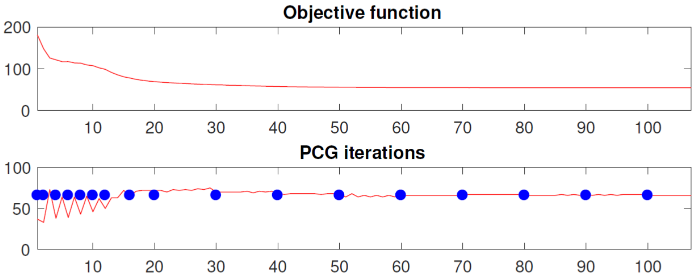
	Forcing term A			Forcing term B		
Coefficient	Iterations	Spectral condition	$\dim A_0$	Iterations	Spectral condition	$\dim A_0$
Without preconditioner	>2000	4.5×10^6	—	>2000	4×10^6	—
Two-levels elasticity preconditioner	61	257.2	387	44	140.6	387
Two-levels heat preconditioner	82	374.8	387	58	140.8	387
Two-levels heat random preconditioner	90	388.2	387	69	276.5	387
Two-levels elasticity random preconditioner	72	141.2	387	63	141.1	387

TableResults for the elasticity problem with high contrast coefficient $\kappa = 1 \times 10^{-6}$

Topology

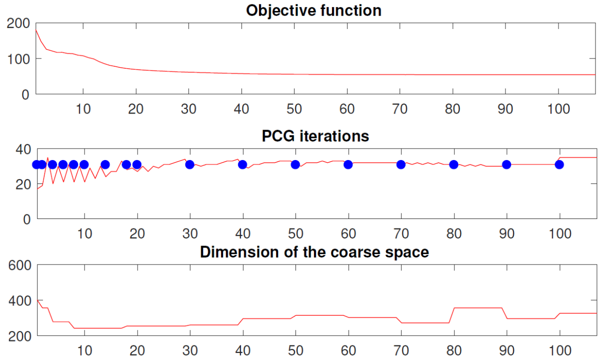


Results



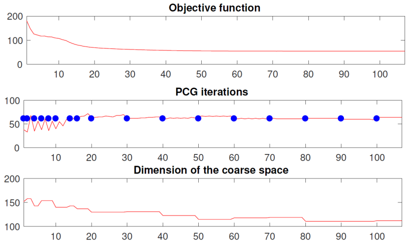
Evolution of parameters in the topology optimization with a two levels elasticity preconditioner.

Results

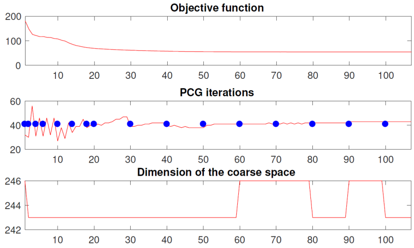


Evolution of parameters in the topology optimization using a preconditioner with heat basis in the coarse space.

Results



Evolution of parameters in the topology optimization using a random preconditioner with elasticity basis.



Evolution of parameters in the topology optimization using a random preconditioner with heat basis.

Conclusions and references

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- Topology optimisation of manufacturable microstructural details without length scale separation using a spectral coarse basis preconditioner J Alexandersen, BS Lazarov Computer Methods in Applied Mechanics and Engineering 290, 156-182

- Generalized multiscale finite element methods (GMsFEM) Y Efendiev, J Galvis, TY Hou Journal of Computational Physics 251, 116-135

- Domain decomposition preconditioners for multiscale flows in high contrast media J Galvis, Y Efendiev Multiscale Model. Simul. 8 (4), 1461-1483