Successive Iteration of Analysis and Design (SIAD) for large-scale eigenvalue topology optimization

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Background: Large scale eigenvalue-related TO

- Natural frequency/ forbidden interval/ frequency gap
- PnC and PtC metamaterials
- Linear buckling

**Accurate analysis requires fine FE discretization.**

**Example:** frequency maximization

\[
\begin{align*}
\text{max :} & \quad \omega_j \\
\text{s.t.} & \quad K\phi_j = \omega_j^2 M\phi_j \\
& \quad \sum_{e=1}^{N_e} x_e V_e - f_y V_0 \leq 0 \\
& \quad 0 < x \leq x_e \leq 1, \quad e = 1, \ldots, N_e \\
\end{align*}
\]

RAMP model \( E_e = \frac{x_e}{1 + p(1-x_e)} E_0, \quad \rho_e = x_e \rho_0 \)
Major issue: computational cost for analysis

<table>
<thead>
<tr>
<th>Procedures</th>
<th>Computing time per iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assembling global stiffness matrices</td>
<td>33 sec. (3.1%)</td>
</tr>
<tr>
<td>Finite element solution $Ku = f$</td>
<td>1005 sec. (95.5%)</td>
</tr>
<tr>
<td>Sensitivity analysis &amp; optimization</td>
<td>14 sec. (1.3%)</td>
</tr>
</tbody>
</table>

Cost for **static compliance minimization** of a 12.83 Million DOF model

For **eigenvalue-related** problems, the difficulty is even worsened by higher-order eigenvalues, and **repeated eigenvalues**.
Conventional (nested double-loop) approach

Initial trial eigenmode

Eigenmodes computed with iterations

Expensive nested Inner-loop

\[ K\phi_j = \omega_j^2 M\phi_j \]

Outer-loop for design updating
Basic idea of SIAD method

Single loop iteration for eigenvalue analysis and design

Approximate eigenmodes improved through a linear equation

Kang et al. *CMAME* 2020; 362: 112847
Procedures of SIAD

- Case of fundamental frequency

\[ K^k \tilde{\phi}_1^k = M^k \tilde{\phi}_1^{k-1} \]

- “Inverse iteration” (single step)
- Similar cost as in static FEA

\[ \tilde{\omega}_1^k = \sqrt{\left( \tilde{\phi}_1^k \right)^T K^k \tilde{\phi}_1^k} \]

Approx. mode: \( \tilde{\phi}_1^{k-1} \)

Improved approx. mode: \( \tilde{\phi}_1^k \)

Design updating

\[ k = k + 1 \]

Approx. eigenfrequency (Rayleigh quotient)

Approx. Sensitivity

\[ \frac{\partial \tilde{\omega}_1^k}{\partial x_e} = \frac{\left( \tilde{\phi}_1^k \right)^T \left( \frac{\partial K^k}{\partial x_e} - \left( \tilde{\omega}_1^k \right)^2 \frac{\partial M^k}{\partial x_e} \right) \tilde{\phi}_1^k}{2\tilde{\omega}_1^k} \]
Discussion on convergence (case of distinct eigenvalue)

Exact eigenpairs for $(k-1)$th design: $\lambda_1^{k-1}, \lambda_2^{k-1}, ..., \lambda_n^{k-1}$, $\phi_1^{k-1}, \phi_2^{k-1}, ..., \phi_n^{k-1}$

for $k$th design: $\lambda_1^k, \lambda_2^k, ..., \lambda_n^k$, $\phi_1^k, \phi_2^k, ..., \phi_n^k$

\[
\Delta \lambda_j^k = (\phi_j^{k-1})^T (\Delta K^k - \lambda_j^{k-1} \Delta M^k) \phi_j^{k-1}
\]

\[
2 (\phi_j^{k-1})^T M^{k-1} \Delta \phi_j = - (\phi_j^{k-1})^T \Delta M^k \phi_j^{k-1}
\]

Approximate eigenpairs for $(k-1)$th and $k$th designs: \(\tilde{\phi}^{k-1} = \sum_{j=1}^{n} a_j^{k-1} \phi_j^k\), \(\tilde{\phi}^{k-1} = \sum_{j=1}^{n} a_j^{k-1} \phi_j^{k-1}\)

Define: $\eta_j^{k-1} = \frac{a_j^{k-1}}{a_1^{k-1}}$, $j = 1, 2, ..., n$ \(\eta_j^k = \frac{\lambda_1^k a_j^k}{\lambda_j^k a_1^k}\), $j = 1, 2, ..., n$

We prove that convergence can be ensured on the condition that design change in each iteration is sufficiently small:

\(\eta_j^k \approx \frac{\lambda_1^{k-1} a_j^{k-1}}{\lambda_j^{k-1} a_1^{k-1}} < \eta_j^{k-1}\), $j = 2, 3, ..., n$

CMAME 2020; 112847
Case of higher-order frequencies

Subspace iteration-like procedures:

\[
K^k \tilde{\Psi}^k = M^k \tilde{\Psi}^{k-1}
\]

Inverse iteration (one step) of subspace

\[
(K^*)^k \tilde{A}^k = (M^*)^k \tilde{A}^k \tilde{\Lambda}^k
\]

Eigenpairs in subspace (low-dimensional)

\[
\tilde{\Psi}^k = \tilde{\Psi}^k \tilde{A}^k
\]

Mode updating

* For repeated eigenvalues, a sub-problem of eigenvalue analysis needs to be solved to find Gateaux differentiable eigenmodes:

\[
F_c = \lambda_i' c_i \\
F = (\tilde{\Phi}^k)^T \left( \partial K^k / \partial x - (\tilde{\omega}_r^k)^2 \partial M^k / \partial x \right) \tilde{\Phi}^k
\]

\[
\tilde{\Phi}^k = \Phi^k C
\]
Example: Fundamental frequency maximization

Mesh: 192 × 64 × 64
DOFs: 2,446,275

<table>
<thead>
<tr>
<th>Method</th>
<th>Number of full-scale equation solutions</th>
<th>Computing time (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIAD</td>
<td>100</td>
<td>22,980</td>
</tr>
<tr>
<td>Double-loop</td>
<td>1058</td>
<td>247,680</td>
</tr>
</tbody>
</table>

SIAD 3.97 rad/s
Double-loop 4.08 rad/s
Insensitive to initial trial eigenvectors

Iter 1:
- Random: 1.7676 rad/s
- Cosine: 12.9885 rad/s
- All-one: 1.7734 rad/s

Iter 30:
- Random: 2.7582
- Cosine: 2.6342
- All-one: 2.7578

Iter 50:
- Random: 3.7638
- Cosine: 3.7406
- All-one: 3.7639

Iter 70:
- Random: 3.8515
- Cosine: 3.8519
- All-one: 3.8513
Example: Fundamental frequency maximization

Mesh: $160 \times 160 \times 80$
DOFs: 6,298,803

Optimized design obtained with random initial trial eigenmodes

Approx. eigenmode converges quickly.
Example: frequency gap maximization

\[
\begin{align*}
\text{max : } & \quad \omega_2 - \omega_1 \\
\text{s.t. } & \quad \omega_3 - \omega_2 \geq \beta \\
& \quad \sum_{e=1}^{N_e} x_e V_e - f_v V_0 \leq 0 \\
& \quad 0 < x \leq x_e \leq 1, \quad e = 1, \ldots, N_e
\end{align*}
\]

Mesh: 256x64x64
DOFs: 3,257,475

Repeated eigenvalues: \( \omega_3 = \omega_4 \)
Conclusions

- SIAD implements **concurrent convergence** of eigenvalue analysis and design optimization.
- Affordable solution of eigenvalue optimization with **millions of DOFs** on a desktop computer.

THANK YOU!